Green's function for a generalized two-dimensional fluid

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Introduction

Generalized two-dimensional (2D) fluid

The Governing equation of this system is the nonlinear advection equation for a scalar q, including a real parameter α.

$$\frac{\partial q}{\partial t} + J(\psi, q) = \mathcal{D} + \mathcal{F}$$
$$\hat{q}(\boldsymbol{k}) = -|\boldsymbol{k}|^{\alpha}\hat{\psi}(\boldsymbol{k})$$

(Pierrehumbert *et al.* 1994)

- $q(\boldsymbol{r}, t)$: "vorticity"
- $\psi(\boldsymbol{r}, t)$: stream function
- $J(a, b) \equiv \partial_x a \partial_y b - \partial_x b \partial_y a$
- $\hat{a}(\boldsymbol{k})$: Fourier transform of $a(\boldsymbol{r}, t)$
- $\mathcal{D}$: dissipation
- $\mathcal{F}$: forcing

The above equation reduces to well known governing equations of geophysical 2D fluids.

- $\alpha = 2$: the vorticity equation for the Navier-Stokes system
- $\alpha = 1$: the thermodynamic energy equation for the surface quasi-geostrophic system (Held, *et al*. 1995)
- $\alpha = -2$: the quasi-geostrophic potential vorticity equation in the limit of small deformation radius for the shallow water system on an f-plane

Motivation to study on the generalized 2D fluid

- To understand several geophysical 2D fluids from unified points of view
- To understand the NS system more deeply
  - Investigate universality and peculiarity of the NS system compared to the other 2D fluids

Previous studies on the generalized 2D fluid

- Homogeneous and isotropic turbulence
  - Enstrophy inertial range (transition of the exponent of enstrophy spectrum $Q(k)$)
    - Pierrehumbert *et al*. (1994), Schorghofer(2000), Watanabe and Iwayama (2004, 2007)

$$Q(k) \sim \begin{cases} k^{-(7-2\alpha)/3} & (0 < \alpha < 2) \\ k^{-1}\ln k & (\alpha = 2) \\ k^{-1} & (2 < \alpha) \end{cases}$$

enstrophy $\mathcal{Q} \equiv \frac{1}{2}\int q^2 \, \mathrm{d}\boldsymbol{r} \equiv \int Q(k)\,\mathrm{d}k$

  - Energy inertial range
    - Smith *et al*. (2002)
- Anisotrophic turbulence
  - Sukhatme and Smith (2009)
- Dual cascade processes
  - Tran (2004), Gkioulekas and Tung (2007)

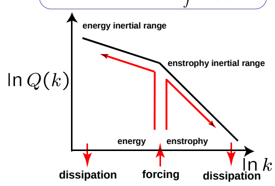

Fig. 1: Schematic dual cascading spectrum of forced generalized 2D turbulence

Purpose of the present study

- We discuss the Green's function for the generalized 2D fluid.
  - We refer the Green's function as the stream function generated by the delta functional distribution of the vorticity $q(\boldsymbol{r}) = \delta(\boldsymbol{r})$.
  - It is well known that the Green's function includes fundamental properties of flows generated by the vorticity.
  - Up to now, the Green's functions, except for $\alpha = 1$ and 2, have not been discussed yet.
- In particular, we pay our attentions on the following three points:
  1. Does the functional form of the Green's function continuously change with the change of α ?
  2. Whether physically reasonable 2D fluids exist for all values of α ?
  3. Is it possible to explain the peculiar turbulent property of the generalized 2D fluid (the existence of the transition point at $\alpha = 2$) in terms of the Green's function ?

Formulation

- We calculate the inverse Fourier transform of $\hat{G}(\boldsymbol{k}) = -|\boldsymbol{k}|^{-\alpha}\hat{\delta}(\boldsymbol{k})$.
- Suppose that
  - the flow domain is an infinite plane,
  - the point vortex is placed at the origin of the coordinate.
- Then we obtain the integral

$$G(r) = -\frac{1}{2\pi}\int_0^1 \mathrm{d}z \left\{ \frac{1}{\sqrt{1-z^2}} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ikzr}}{|k|^{\alpha-1}} \right\}$$

- We calculate the above integral using the Fourier transform and the Gamma function.

Results and Discussion

I. Green's function

The functional form of the Green's function depends on α.

1. Except for particular values of α, where $\alpha \neq \pm 2n$ and n being the integers,

$$G(\boldsymbol{r}) = \Psi(\alpha) r^{\alpha-2}$$
$$\Psi(\alpha) = -\frac{1}{2^{\alpha}\pi}\frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}$$

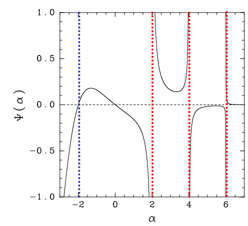

Fig. 2: Dependence of the coefficients $\Psi(\alpha)$ on α

2. For α being the positive even numbers, $\alpha = 2m$, where m is the natural numbers,

$$G(\boldsymbol{r}) = \Psi_{\mathrm{L}}(\alpha) r^{\alpha-2}(\ln|\boldsymbol{r}| + C)$$
$$\Psi_{\mathrm{L}}(\alpha) = \frac{(-1)^{(\alpha-2)/2}}{2^{\alpha-1}\pi\{\Gamma(\frac{\alpha}{2})\}^2}$$

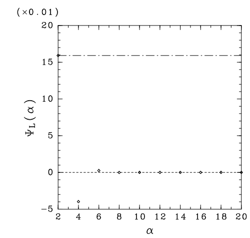

Fig. 3: Dependence of the coefficients $\Psi_{\mathrm{L}}(\alpha)$ on α

3. For α being the negative even numbers, $\alpha = -2n$,

$$G(\boldsymbol{r}) = (-1)^{\frac{|\alpha|}{2}+1}\Delta^{\frac{|\alpha|}{2}}\delta(\boldsymbol{r})$$

II. Azimuthal velocity around the vortex

The azimuthal velocity around the vortex can be calculated by differentiating the Green's function. For $\alpha \neq \pm 2n$,

$$u = \frac{\partial G(\boldsymbol{r})}{\partial r} = (\alpha - 2)\Psi(\alpha) r^{\alpha-3}$$

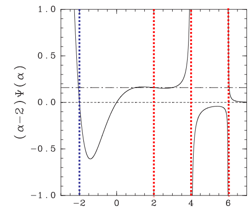

Fig. 4: Dependence of the coefficient $(\alpha-2)\Psi(\alpha)$ on α

III. Existence of physically reasonable 2D fluids

- The azimuthal velocity around the vortex is a increasing function of r for $\alpha > 3$.
  - It is physically reasonable that the azimuthal velocity decreases as the distance from the velocity source being farther.
- We conclude that physically reasonable 2D fluids exist only for $\alpha \leq 3$.

Conclusion

- We have discussed the Green's function for the generalized two-dimensional fluid.
  - The functional form of the Green's function is

$$G(\boldsymbol{r}) \sim r^{\alpha-2}, \ (\alpha \neq \pm 2n)$$
$$G(\boldsymbol{r}) \sim r^{\alpha-2}\ln r, \ (\alpha = 2m)$$
$$G(\boldsymbol{r}) \sim \Delta^{\frac{|\alpha|}{2}}\delta(\boldsymbol{r}), \ (\alpha = -2n)$$

  - The functional form of the Green's function is discontinuous at $\alpha = \pm 2n$.
  - In contrast, the azimuthal velocity is continuous at $\alpha = 2$.
- Physically reasonable two-dimensional fluids exist only for $\alpha \leq 3$.

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