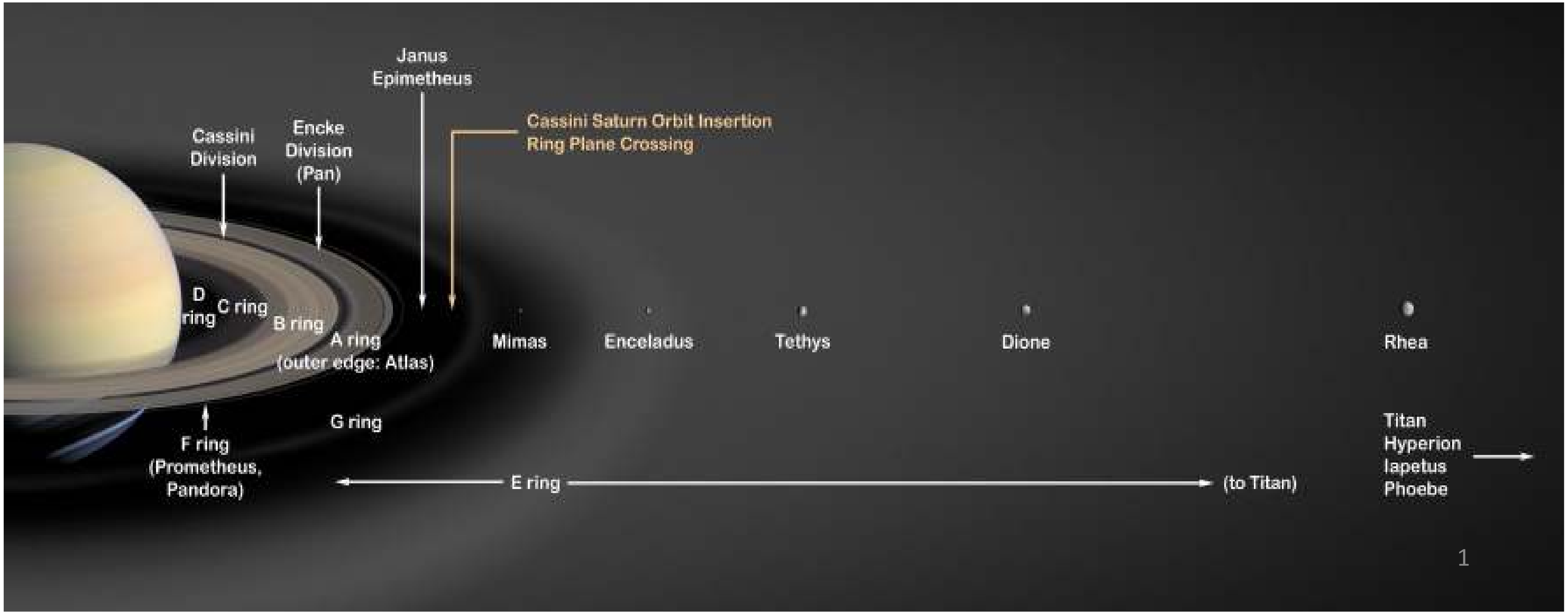
Weakly Nonlinear Analysis of Granular Shear Flow

Kuniyasu Saitoh and Hisao Hayakawa

YITP, Kyoto University, Japan



Contents

Ⅰ.
Shear Band Formation of Sheared Granular Flow

Introduction – *"What's Granular Particle?"*

Inelastic Collision

Restitution coefficient

$$0 < e < 1$$

material constant?

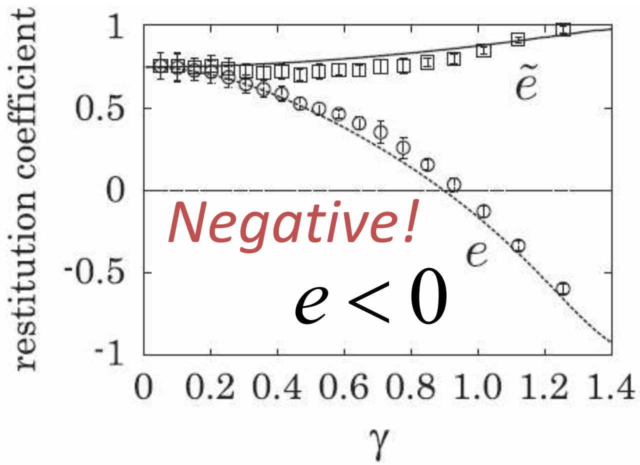
Is that true?

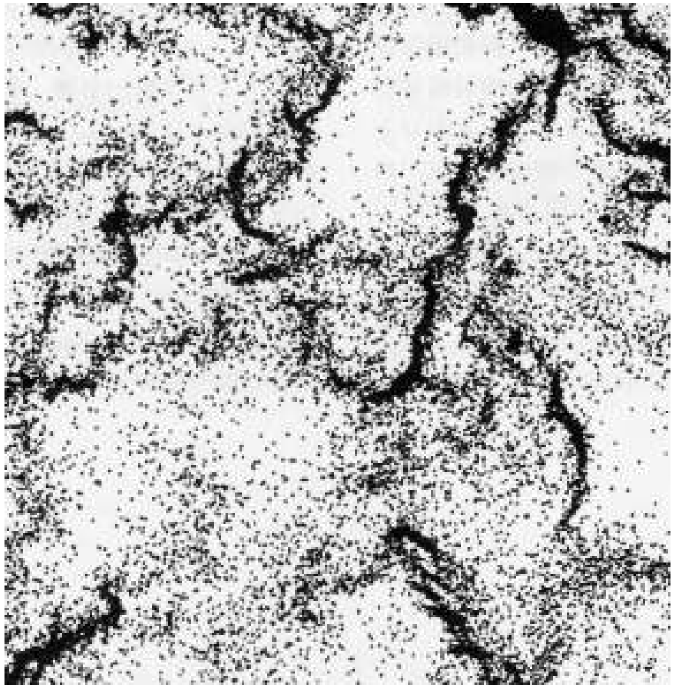
Depends on *Impact Speed, Situation, Size*

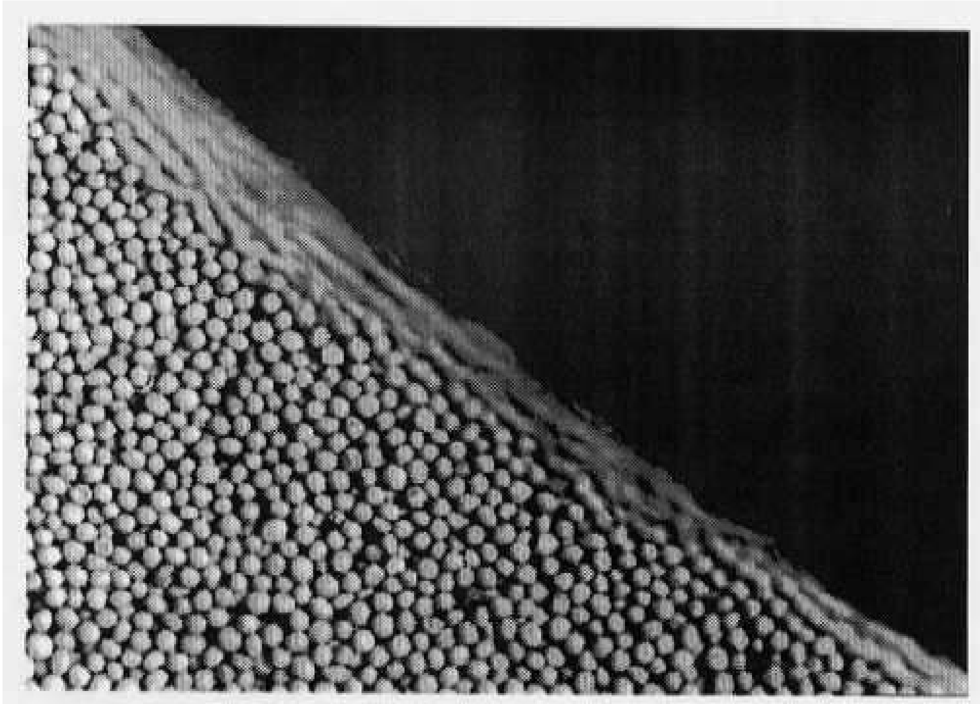
Elasticity, Surface tension, viscoelasticity

<u>Negative restitution coefficient is found in oblique impact of nanoclusters</u>

accepted to Phys. Rev. Lett. (2010)
→ポスター発表





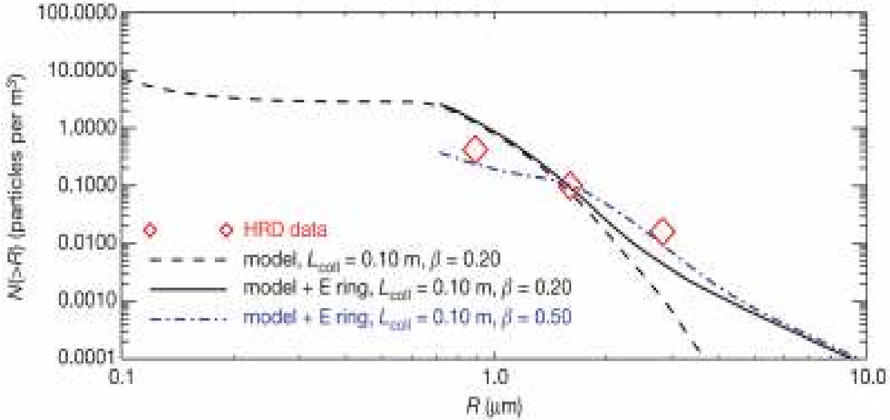


- Many-body system of Dissipative Particles
- Always Non-equilibrium State
- Coexistence of Static Region
- Micro-polar Fluids (Rotation)

Proto-planetary Disk

- Self-Gravity
- Dilute System
- Inelastic Collisions
- Shear Flow

Size Distribution in E-ring



J. Schmidt, N. Brilliantov, F. Spahn, and S. Kempf, Nature **451**, 685 (2008).

Kinetic Theory

Boltzmann eq. for dissipative particles

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = \sigma^2 \int d\mathbf{u}_1 \int d\Omega \Theta(\mathbf{v}_r \cdot \mathbf{n}) \left[\left(\frac{1}{e^2} \right) f'' f_1'' - f f_1 \right]$$

Energy → *not a collision invariant, energy loss rate*

Velocity distribution function

Enskog method, Grad expansion method

$f^{(0)}$ → *Dependent on the system*

Rotational degrees of freedom can be absorbed in e

$$e \simeq \bar{e} - \mu + 2\mu^2 (1 + \bar{e})$$

Hydrodynamic Equations

[Jenkins and Richman (1985), Saitoh and Hayakawa (2007)]

"Jenkins & Richman" without rotation

Area Fraction ν *Velocity Fields* $\mathbf{u}$

Granular Temperature $\theta \equiv \frac{m}{2n}\left\langle (\mathbf{c}-\mathbf{u})^2 \right\rangle$

(scaling units are shown in the next slide)

$$(\partial_t + \mathbf{u}\cdot\nabla)\,\nu = -\nu\nabla\cdot\mathbf{u}$$

$$\nu\,(\partial_t + \mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla\mathbf{P}$$

$$(\nu/2)\,(\partial_t + \mathbf{u}\cdot\nabla)\,\theta = -\mathbf{P}:\nabla\mathbf{u} - \nabla\cdot\mathbf{q} - \chi$$

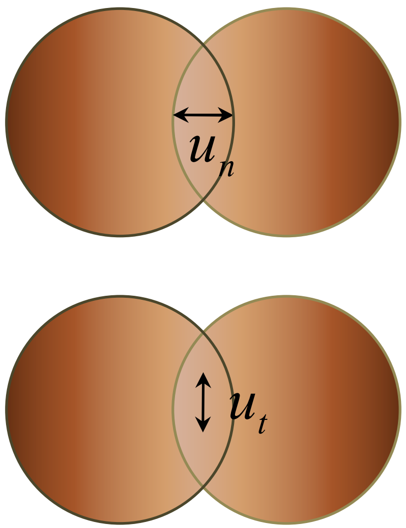
$$P_{ij} = \left[p(\nu)\theta - \xi(\nu)\theta^{1/2}\frac{\partial u_j}{\partial x_j}\right]\delta_{ij} - \eta(\nu)\theta^{1/2}D'_{ij}$$

$$q_i = -\kappa(\nu)\theta^{1/2}\frac{\partial\theta}{\partial x_i} - \lambda(\nu)\theta^{3/2}\frac{\partial\nu}{\partial x_i}$$

$$\chi = \frac{1-e^2}{4\sqrt{2\pi}}\nu^2 g(\nu)\theta^{1/2}\left[4\theta - 3\sqrt{\frac{\pi}{2}}\theta^{1/2}\frac{\partial u_j}{\partial x_j}\right].$$

DEM Simulation

[Saitoh and Hayakawa (2007)]

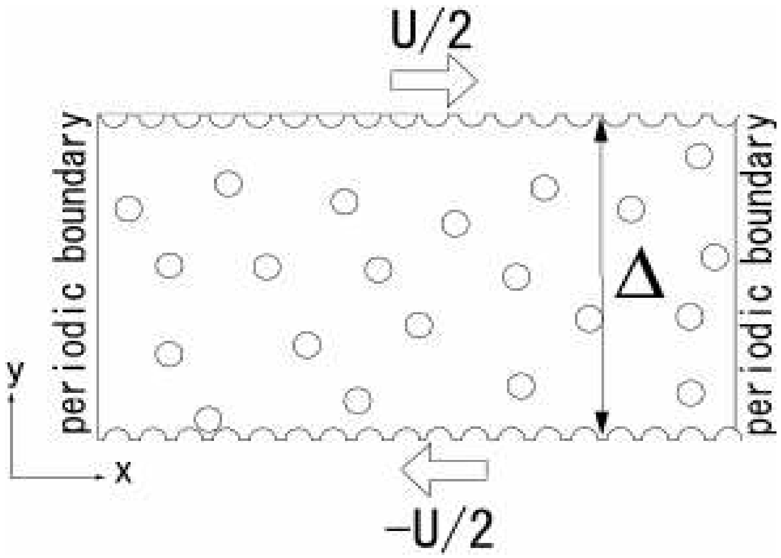


Normal Force

$$F_n = -k_n u_n - \eta_n \frac{du_n}{dt}$$

Tangential Force

$$F_t = \begin{cases} -k_t u_t - \eta_t \dfrac{du_t}{dt} & \text{If } |F_t| < \mu |F_n| \\ -\mu |F_n| & \text{If } |F_t| \geq \mu |F_n| \end{cases}$$



Setup

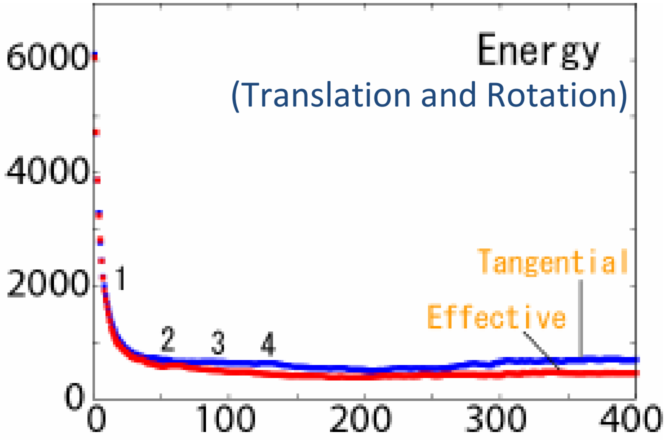
- X-axis : Periodic boundary
- Y-axis : Bumpy boundary
- Scaling Units

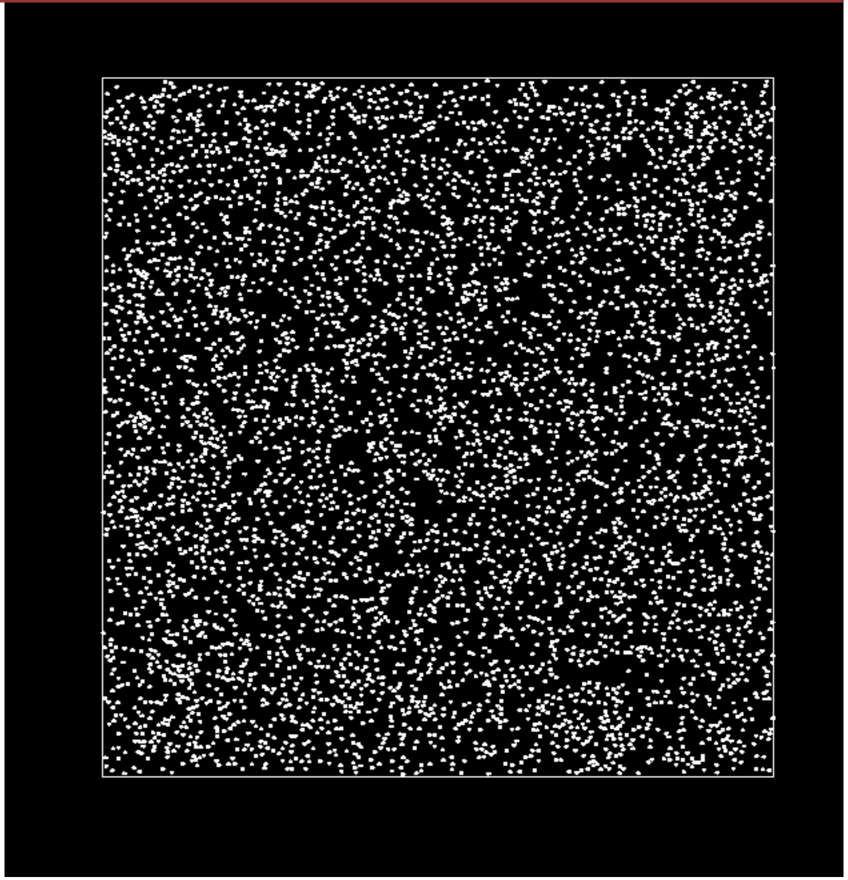
Mass : m
Length : d
Time : $2d / U$

Shear Band Formation

Results

- Initial Condition
  - Configuration is at Random
  - Gaussian Velocity distribution
- Transient Dynamics
  - Two narrow Shear Bands
  - Slowly moves to the center region
  - Collide and Merge
- Steady State
  - A wide Shear Band





- Restitution coefficient : e = 0.85
- Friction constant : μ = 0.2
- The number of particles : N = 5000

Shear Band Formation (Analysis)

Hydrodynamic Fields

Area Fraction $\nu(Y)$

Velocity Fields $\mathbf{u} = (u(Y), w(Y))$

Granular Temperature

$$\theta(Y) \equiv \frac{m}{2n}\sum_{i=1}^{n}(\mathbf{c}_i - \mathbf{u})^2$$

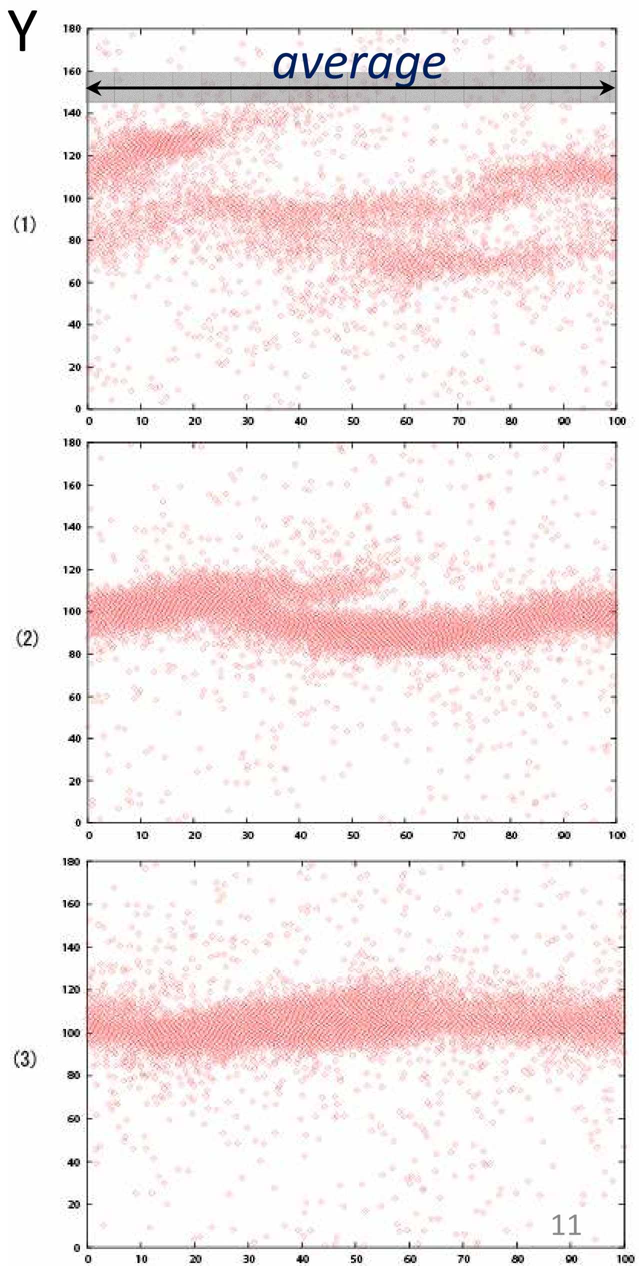
Solve "Jenkins & Richman"

Boundary Conditions

$$-\boldsymbol{n}\cdot\boldsymbol{P}\cdot\boldsymbol{t} = \frac{\pi}{4}\phi\Omega(\nu,\theta)|\boldsymbol{u}_{sl}|$$

$$\boldsymbol{n}\cdot\boldsymbol{q} = -\boldsymbol{u}_{sl}\cdot\boldsymbol{P}\cdot\boldsymbol{n} - \underline{\Gamma(\nu,\theta)}$$

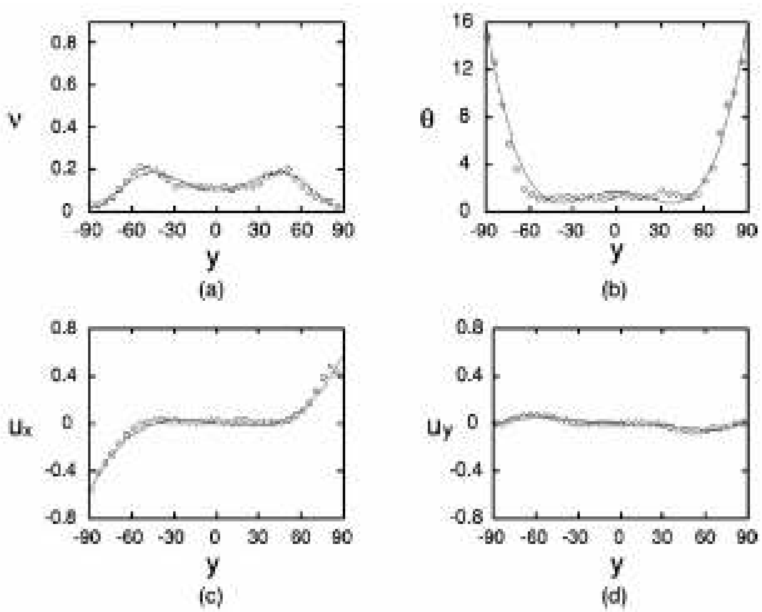
energy loss rate



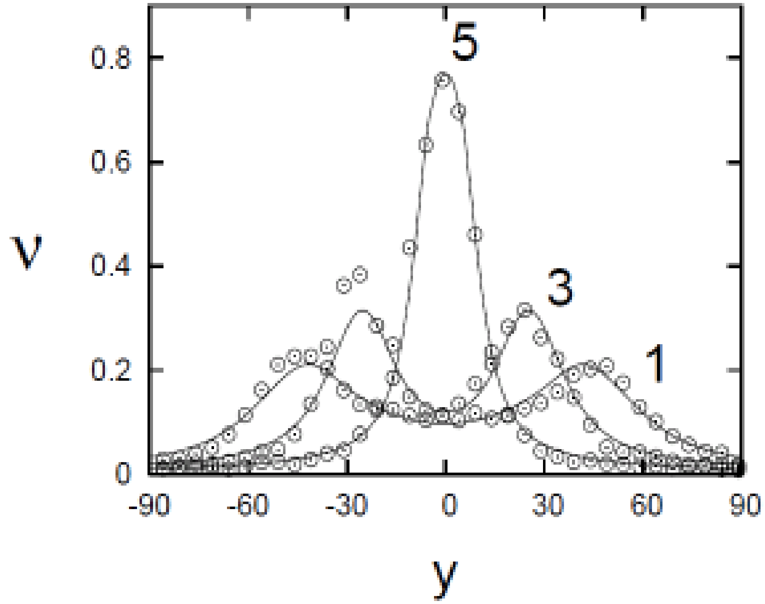
Shear Band Formation (Analysis)

Results

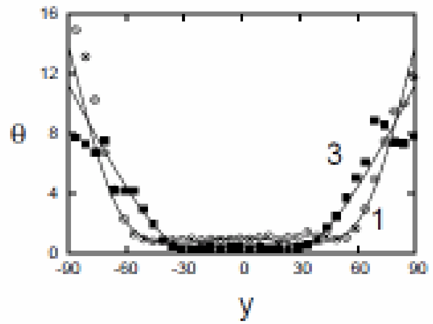
Initial Condition



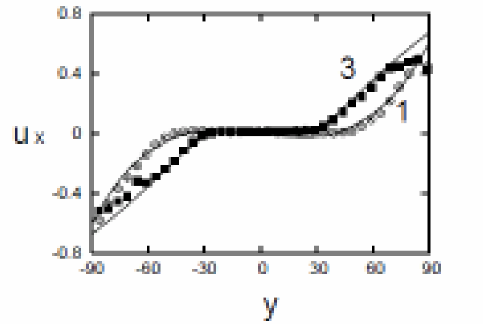
Area Fraction



Granular Temperature

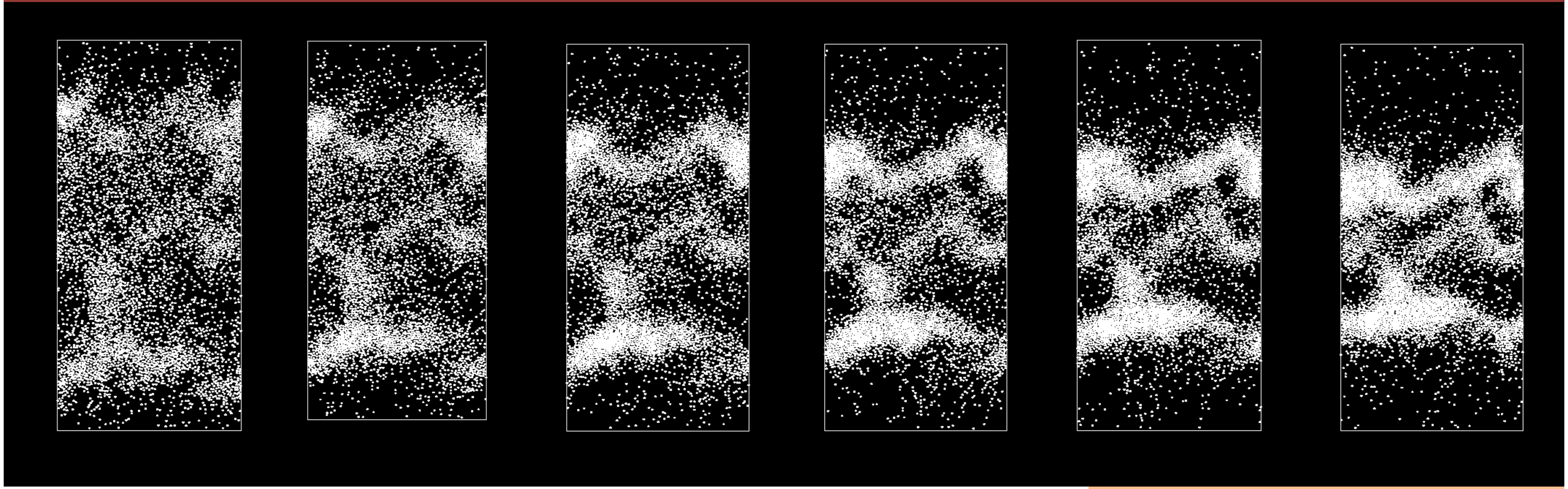


Velocity field (x direction)



- Kinetic Theory well reproduces the results, *even if the density is high*

Analytic Solution for Steady State



● Mean Area Fraction : 0.24

$$\frac{d}{dy}\left\{F(\nu)\frac{d\nu}{dy}\right\}=G(\nu)$$

$$F(\nu)\equiv p(\nu)^{-3/2}\left\{\frac{\kappa(\nu)}{2p(\nu)}\left[1+(1+e)\frac{d(\nu^2 g(\nu))}{d\nu}\right]-\lambda(\nu)\right\}$$

$$G(\nu)\equiv 2\left(\frac{\tau}{p}\right)^2\frac{p(\nu)^{1/2}}{\eta(\nu)}-\frac{(1-e^2)\nu^2 g(\nu)}{\sqrt{2\pi}\,p(\nu)^{3/2}}$$

$$P_{xy}\mid_{boundary}=\tau$$

$$P_{yy}\mid_{boundary}=p$$

- ➢Transient Dynamics is Complex
- ➢Steady State is also well reproduced

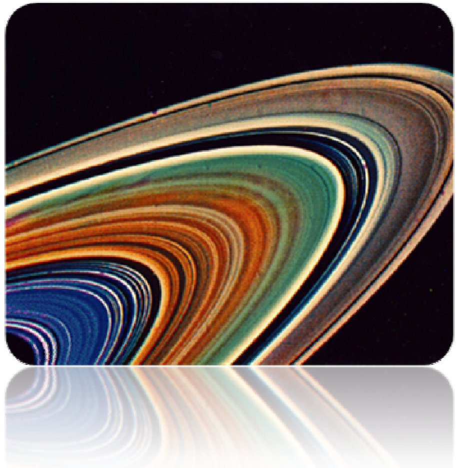
Discussion and Conclusion

Discussion

- More Realistic Models
  ex. Shear Flow of Cohesive Granular Particles
- Influence of Self-Gravity
- Quasi-2D system
- Effects of Jamming and Rotation in the Dense Region
- Fragmentation and Aggregation

Conclusion

- ✓ Shear Band Formation is found in the DEM simulation
- ✓ Analytic Solution of Steady Shear Band
- ✓ Kinetic Theory works well, even if the Density is High



Ⅱ.

Linear stability analysis and Weakly nonlinear analysis of Sheared Granular Flow

Linear Stability Analysis

Growth Rate

$$\phi - \phi_0 = \hat{\phi} \propto e^{\sigma t} \qquad L\hat{\phi} = \sigma \hat{\phi} \qquad \mathrm{Re}[\sigma]$$

Weakly Nonlinear Analysis

Near the Neutral Mode $\sigma = \sigma_c + \alpha \varepsilon^2 + \cdots$

$$\partial_\tau A = \alpha A + \gamma A^3 + \cdots$$

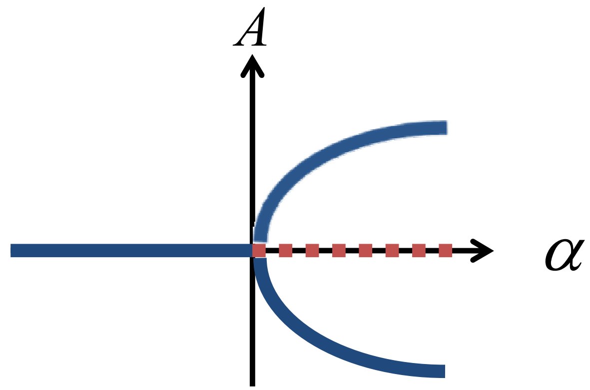
$\sigma_c = 0$ **Pitchfork Bifurcation**, ex. Super-/Sub-critical bifurcation

$\sigma_c = i\omega$ **Hopf Bifurcation**, ex. Limit Cycle

$$\sigma_c = 0$$

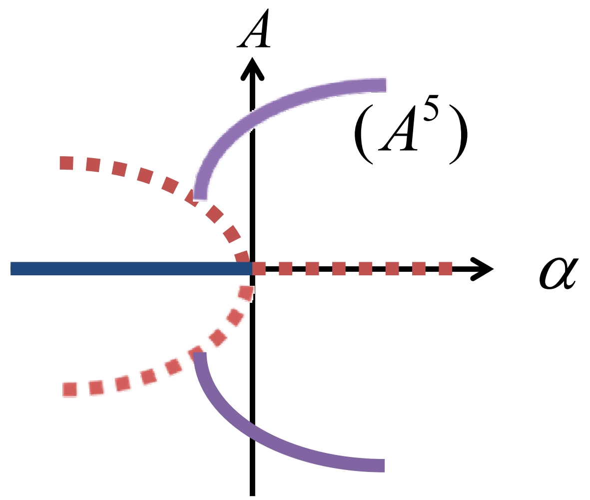
$$\alpha > 0, \quad \gamma < 0$$

Supercritical



$$\alpha > 0, \quad \gamma > 0$$

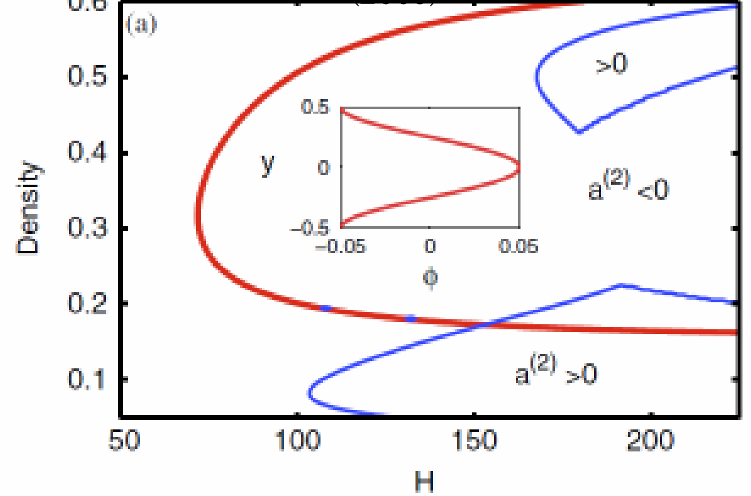
Subcritical



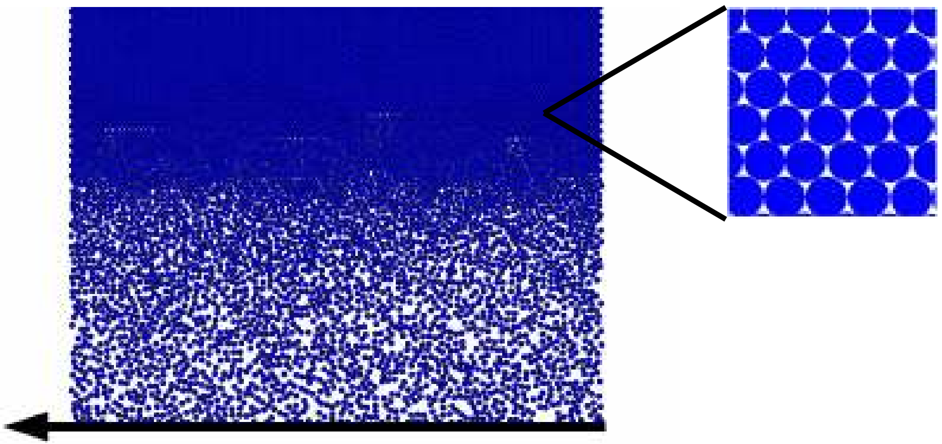
External field, ex. Gravity,
サドルノード分岐

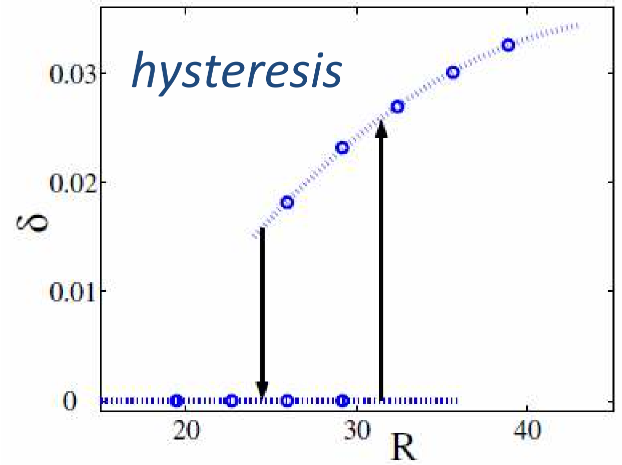
Previous Works

<u>Granular Gas</u> P. Shukla and M. Alam (2009)



<u>Dense System</u> E. Khain (2009)





- *Homogeneous State is Unstable*
- *Granular Gas:*
  - *Supercritical Bifurcation*
  - *Subcritical Bifurcation*
- *Dense System:*
  - *Subcritical Bifurcation*

縮約によるアプローチ

（高密度な場合を除き）粉体集団のダイナミクスを記述する流体力学的な方程式

先行研究

P. Shukla and M. Alam (2009)

Reynolds & Potter (1967)の手続き

↓

Stuart-Landau型の振幅方程式

本研究

Newell & Whitehead (1969)の手続き

※Y. Kuramoto, “Chemical Oscillations, Waves, and Turbulence”

↓

Ginzburg-Landau方程式

問題点

- せん断方向の依存性を無視
- 長さスケールが残らない
- 摂動に特別な境界条件を設定

特長

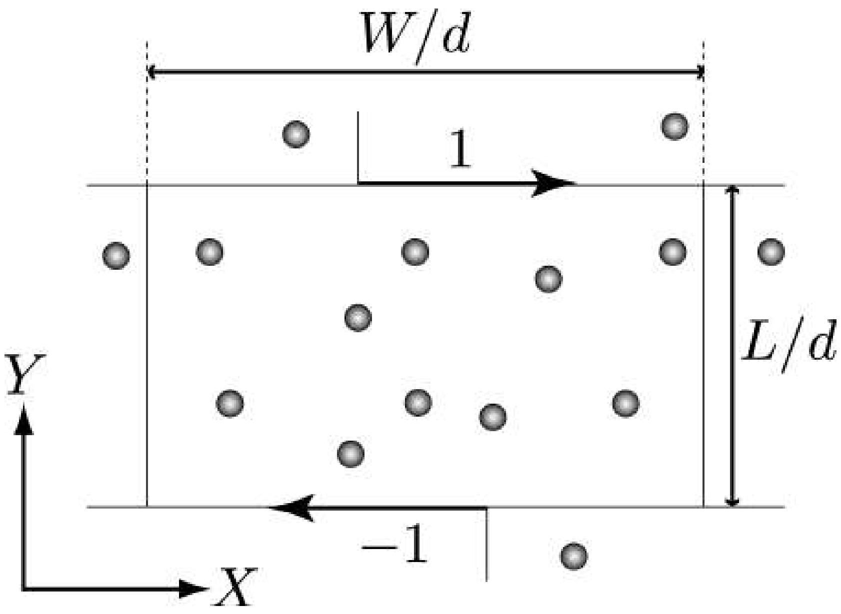
- ✓せん断方向の依存性を考慮
- ✓長さスケールを残し空間構造を捉える
- ✓Lees-Edwards境界を適用

Setup

Lees-Edwards Boundary

Shear rate

$$\dot{\gamma} = 2d / L$$



Basic equation

<u>*"Jenkins & Richman"*</u>

Hydrodynamic Field

$$\phi = (\nu, u, w, \theta)$$

Base State

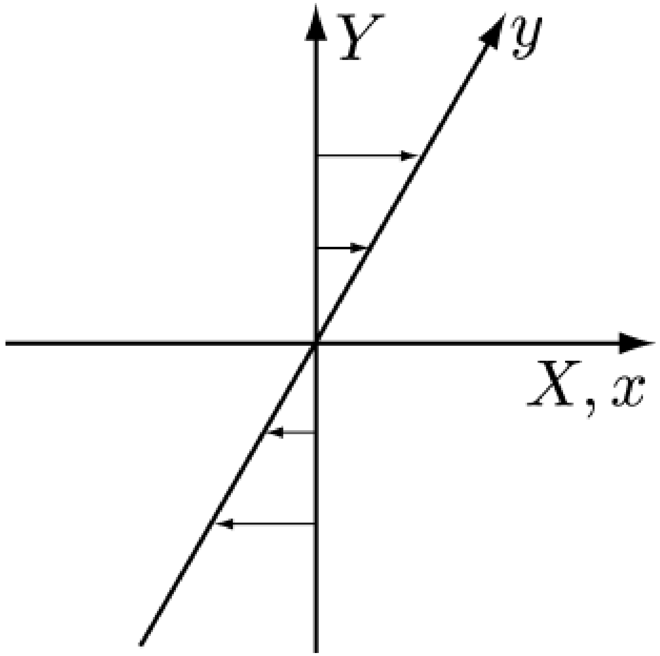
$$\phi_0 = (\nu_0, \dot{\gamma}Y, 0, \theta_0)$$

Convective Term

$$\partial_t \phi + \mathbf{v} \cdot \nabla \phi = \partial_t \phi + \dot{\gamma} Y \partial_X \phi + \ldots$$

Stretched Coordinate

$$\begin{cases} x = X - \dot{\gamma} t Y \\ y = Y \\ \tilde{t} = t \end{cases} \qquad \begin{cases} \partial_X = \partial_x \\ \partial_Y = \partial_y - \dot{\gamma} \tilde{t} \partial_x \\ \partial_t = \partial_{\tilde{t}} - \dot{\gamma} y \partial_x \end{cases}$$



Base State

$$\tilde{\phi}_0 = (\nu_0, 0, 0, \theta_0) \qquad \theta_0 = \sqrt{\frac{\pi}{2}} \frac{\dot{\gamma}^2 \eta(\nu_0)}{(1 - e^2)\, \nu_0^2 g(\nu_0)}$$

Linear equation

$$\hat{\phi} = \phi - \phi_0 \qquad \partial_{\tilde{t}}\hat{\phi} = \mathcal{L}\hat{\phi}$$

Symmetry

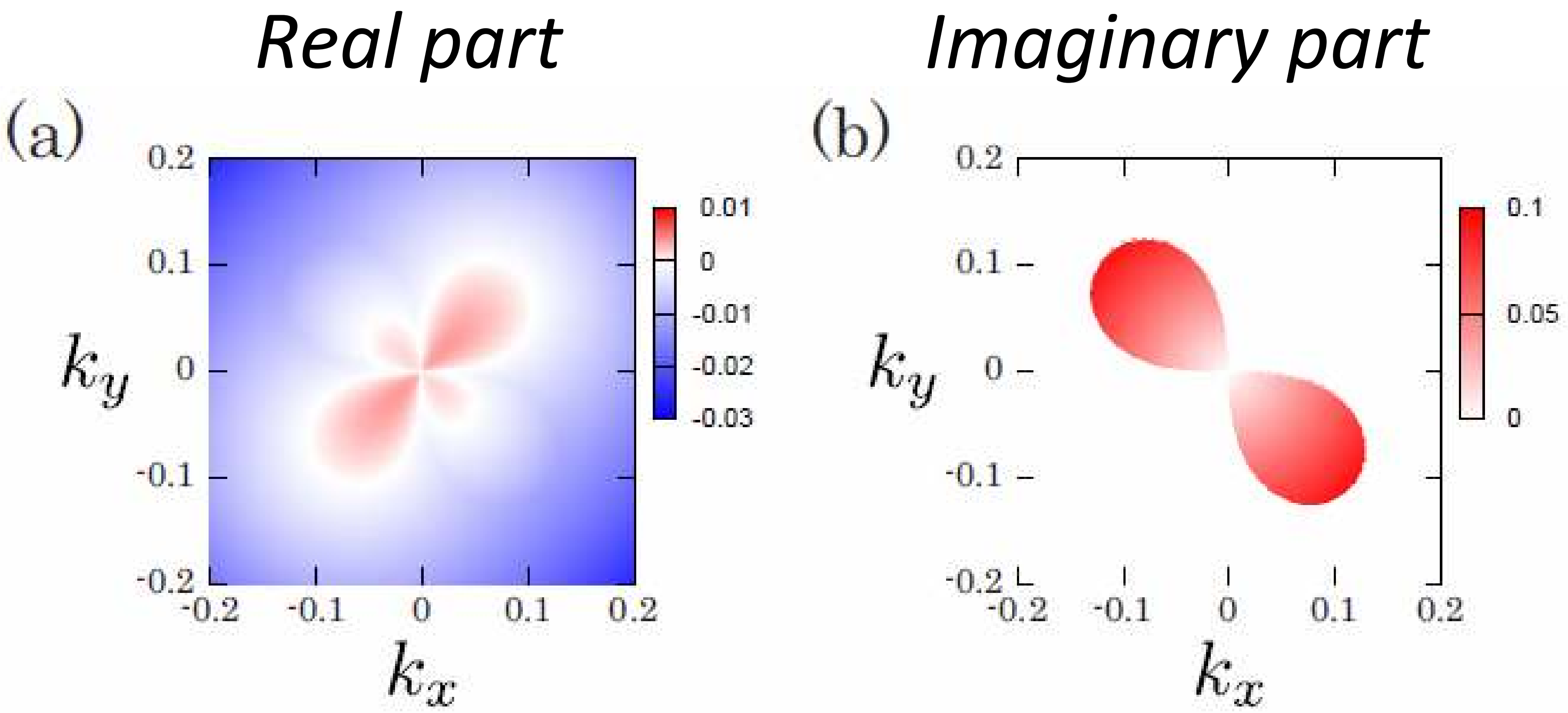
$$\begin{pmatrix} v(-x,-y,\tilde{t}) \\ u(-x,-y,\tilde{t}) \\ w(-x,-y,\tilde{t}) \\ \theta(-x,-y,\tilde{t}) \end{pmatrix} = \begin{pmatrix} v(x,y,\tilde{t}) \\ -u(x,y,\tilde{t}) \\ -w(x,y,\tilde{t}) \\ \theta(x,y,\tilde{t}) \end{pmatrix}$$

Fourier Transform

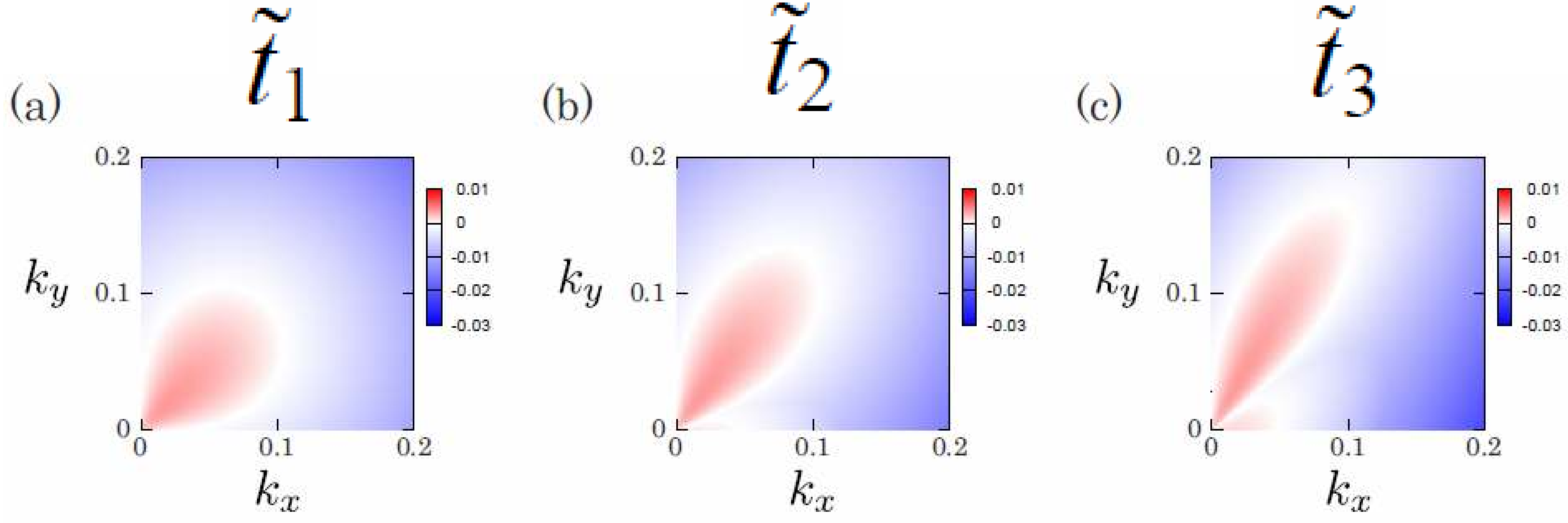
$$\begin{cases} k_x = k_X \\ k_y(t) = k_Y + \dot{\gamma} t k_X \end{cases} \qquad \mathcal{M}\hat{\phi} = \sigma\hat{\phi}$$

↑ *4 × 4 Real Matrix*

Eigenvalue $\tilde{t} = 0$



Linear Stability Analysis



Neutral mode

$$\sigma_c = 0 \longrightarrow \textit{Real number!}$$

$$k_x^c = 2\pi d / L$$

$$k_y^c - \dot{\gamma} t k_x^c = 2\pi d / L$$

Scaling

Hydrodynamic limit

$$L/d \to \infty \quad \textit{i.e.} \quad \dot{\gamma} \to 0$$

Fixed Temperature

$$\theta_0 = (\text{fixed}) \propto \dot{\gamma}^2/(1-e)$$

Expansion Parameter

$$\epsilon^2 \equiv \sqrt{1-e}$$

$$\dot{\gamma} = \epsilon^2$$

Long Length Scale

$$(\xi, \zeta) = \epsilon (x, y)$$

Wave number

$$\mathbf{k} = \epsilon \mathbf{q} \qquad |\mathbf{k}| \sim O(1/x)$$

Long Time Scale … scaled by $\dot{\gamma}$

$$\tau = \dot{\gamma} t = \epsilon^2 t$$

Near the Neutral mode

$$\mathcal{M} = \epsilon \mathcal{M}_1 + \epsilon^2 \mathcal{M}_2 + O\left(\epsilon^3\right)$$

$$\sigma = \epsilon^2 \alpha + O\left(\epsilon^3\right)$$

$$\mathcal{M}\hat{\phi} = \sigma\hat{\phi} \quad \Rightarrow \quad \mathcal{M}_2\hat{\phi} = \alpha\hat{\phi}$$

α determines "Linearly Stable"
or "Linearly Unstable"

Hydrodynamic equations

$$(\partial_{\tilde{t}} - \mathcal{M})\,\hat{\phi} = \mathcal{N}_2\left[\hat{\phi}\hat{\phi}\right] + \mathcal{N}_3\left[\hat{\phi}\hat{\phi}\hat{\phi}\right] + \ldots$$

Disturbance Field

$$\hat{\phi} = \epsilon\phi_1 + \epsilon^2\phi_2 + O\left(\epsilon^3\right)$$

$O(\epsilon^2)$ *Neutral Solution*

$$\phi_1 = \underline{A(\xi,\zeta,\tau)}e^{i\mathbf{q}\cdot\xi} + \text{c.c.}$$

Disturbance Amplitude

Weakly Nonlinear Analysis

$$O(\epsilon^3) \propto e^{i\mathbf{q}\cdot\xi}$$

$$-\mathcal{M}_1\phi_2 = -\partial_\tau\phi_1 + \mathcal{M}_2\phi_1 + \mathcal{D}_1\partial_\xi^2\phi_1 + \mathcal{D}_2\partial_\xi\partial_\mu\phi_1 + \mathcal{D}_1\partial_\mu^2\phi_1 + \mathcal{N}_3\phi_1|\phi_1|^2$$

Zero eigenvector (Left)

$$\phi_a^T\mathcal{M}_1 = 0$$

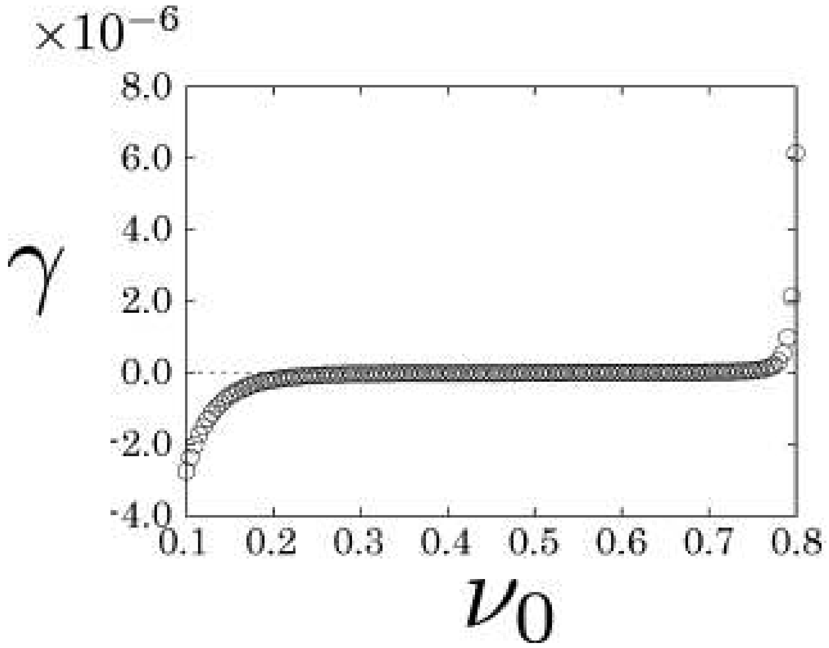
Time Dependent Ginzburg-Landau equation

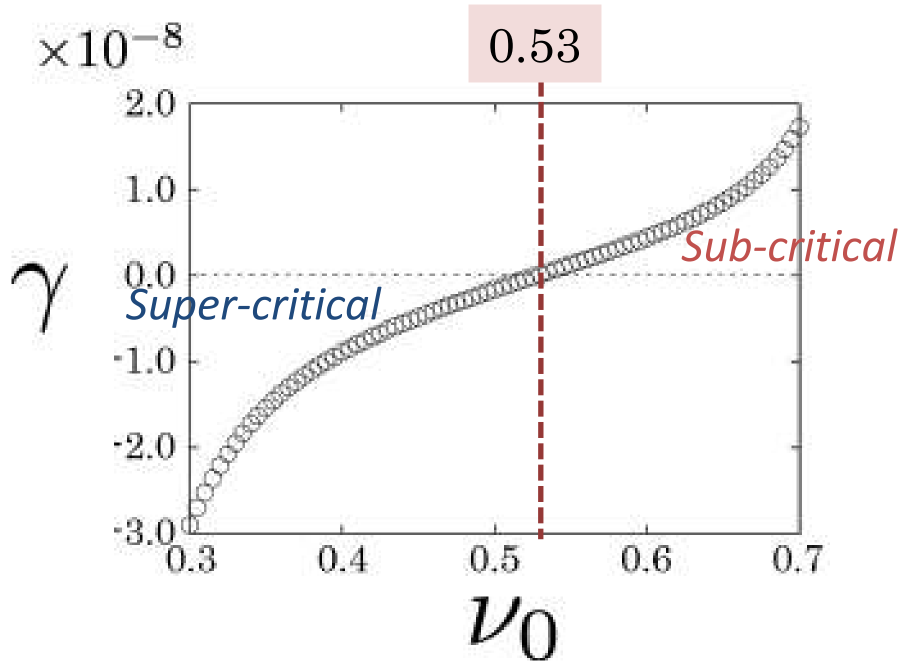
$$\partial_\tau A = \alpha A + D_1\partial_\xi^2 A + D_2\partial_\xi\partial_\mu A + D_1\partial_\mu^2 A + \gamma A|A|^2$$

$$\partial_\mu \equiv \partial_\zeta - \tau\partial_\xi$$

Bifurcation Analysis

Coefficient of $A|A|^2$

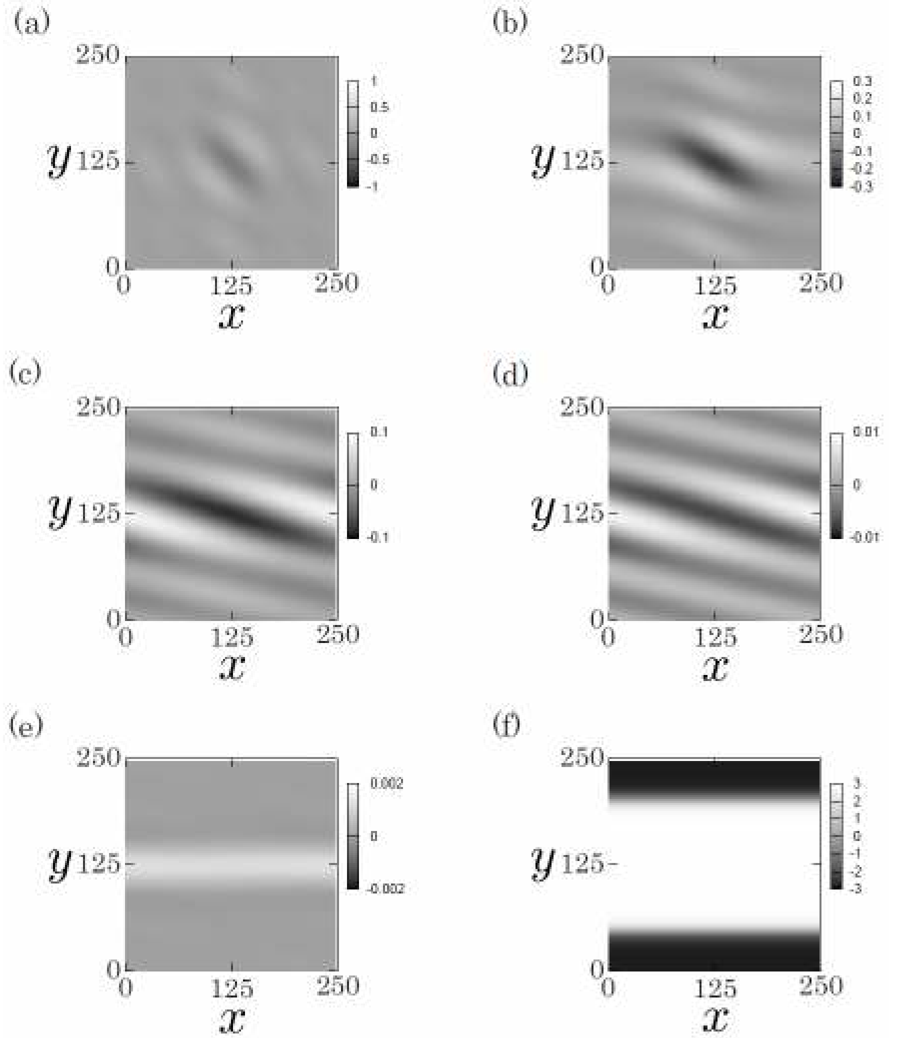




$\varepsilon = 0.1$ $\quad$ $\nu_0 < 0.53$ *Super-critical* ***bifurcation***

$\theta_0 = mU^2/4$ $\quad$ $\nu_0 > 0.53$ *Sub-critical* ***bifurcation***

Super-critical Bifurcation

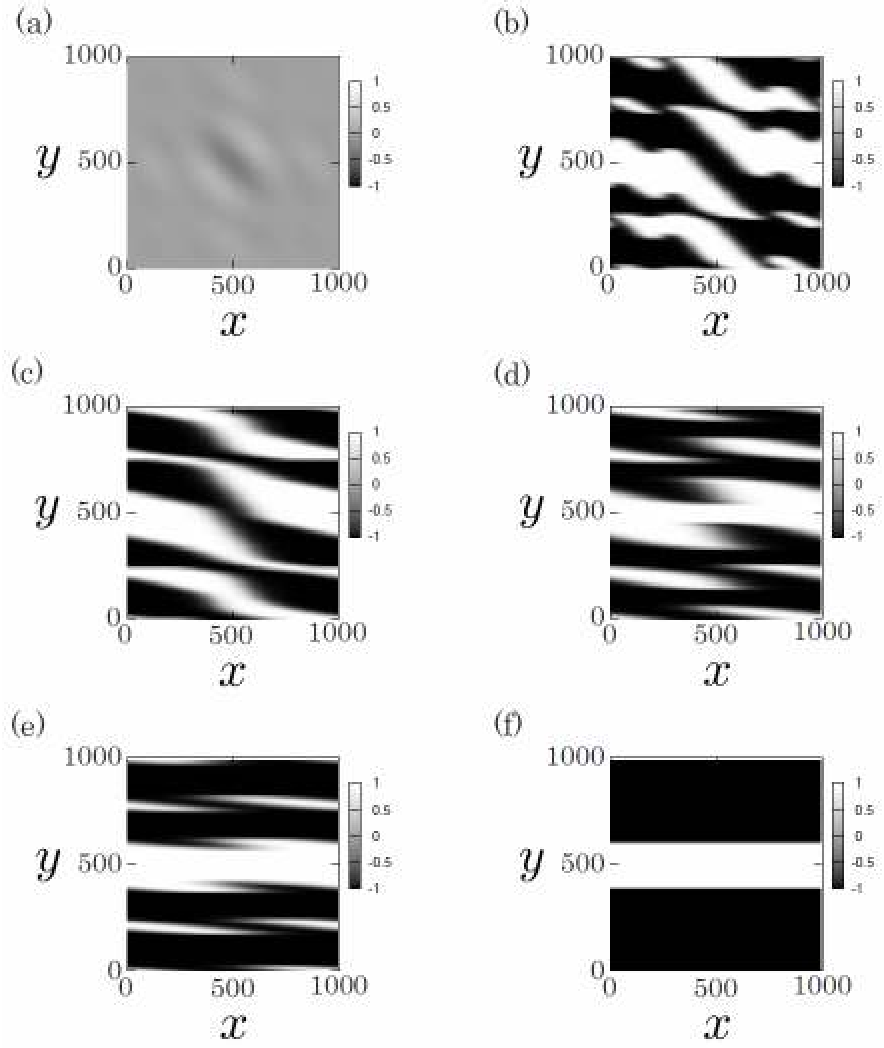


Numerical Analysis of the TDGL equation

Sub-critical Bifurcation

We assume

$$-(\alpha+\gamma)A|A|^4$$



Discussion and Conclusion

Discussion

- Coefficient of $A|A|^4$
- Examine the Sub-critical Bifurcation.

  In Experiment or the DEM simulation

  Bifurcation Parameter U
- Physical Boundary Condition
- Effects of Jamming and Rotation in the Dense Region
- Quasi-2D system

Conclusion

- ✓Derived the TDGL equation for Sheared Granular Flow in the Hydrodynamic limit.

- ✓Sub-critical Bifurcation is found in the Dense Region.

Announcement

京都大学基礎物理学研究所主催　国際研究会

宇宙と物質の非平衡ダイナミクス

日程　：　2011年10月31日～11月3日

場所　：　京都大学　基礎物理学研究所
湯川記念館　Panasonicホール（予定）

（粉体）ジェット　：　Blandford氏, 平野氏, Nagel氏ほか
惑星リング　：　Schmidt氏, Salo氏, Burns氏, Porcco氏ほか
MHD乱流　：　Balbus氏, 坪田氏, 後藤氏ほか